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Separation of variables and exact solution of the Dirac equation in non-static Minkowski spacetimes

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Abstract. In the present paper, using a second-order formalism, a complete separation of variables in the Dirac equation for a free particle in non-static orthogonal curvilinear coordinates of the form $t=f(u, v)$, $x=g(u, v)$, y, z, is presented. It is shown that the Dirac equation is separable in eight non-equivalent coordinate systems where the Klein-Gordon equation separates. Exact solutions of the Dirac equation in the systems of coordinates obtained are presented.

1. Introduction

After the appearance of the classical papers of Stäckel (1897) and Eisenhart (1934), much effort was devoted to the study of the criteria of separability of variables in the Schrödinger, Hamilton, Jacobi and other equations of mathematical physics.

Perhaps the most systematic account of the systems of coordinates where the Klein-Gordon equation is separable in the presence of external electromagnetic fields is given by the Bagrov group (Bagrov *et al* (1982) and references therein) who have tackled the problem using complete sets of commuting differential operators. Another approach to the problem was given by Miller (1977) and Kalnins (1975), using a group theoretical approach and the concept of *R*-separability. They have succeeded in finding new systems of coordinates where the free Klein-Gordon equation is separable.

Further complications arise when we try to analyse the problem of separation of variables in the Dirac equation in flat or curved spacetimes because of its matrix character; in fact we are dealing with a set of four coupled partial differential equations, and there are no general techniques, even in very simple configurations, for solving the problem of finding exact solutions.

The results obtained by Chandrasekhar (1983), showing that the Dirac equation admits separation of variables in oblate spheroidal coordinates, stimulated the search of general results aiming to establish in which systems of coordinates the Dirac equation separates. Here, we have to mention the work of Cook (1982) who, using a modification of the Stäckel method, discusses the problem of separability in the Dirac equation in curvilinear coordinates. The results obtained by Bagrov and their coworkers (1973, 1982) should also be mentioned. They found new exact solutions of the Dirac equation in the presence of electromagnetic fields in curvilinear coordinates obtained from the analysis of the Klein-Gordon equation. Finally we mention the most recent results

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obtained by Kalnins *et al* (1986) studying the Dirac equation, using for this purpose a null tetrad formalism.

Recently, using a further modification of the method of separation of variables proposed by Shishkin (Shishkin 1976, Andrushkevich and Shishkin 1987) some exact solutions to the Dirac equation in curvilinear coordinates have been reported (Shishkin and Villalba 1989, Villalba 1990), showing in this way that the Dirac equation is separable in all the cylindrical coordinate systems where the Laplace equation separates (cylindrical, parabolic cylinder and elliptical cylinder coordinates). The results obtained in static curvilinear coordinates can be generalized for curvilinear coordinates in Minkowski spacetimes, and this is the purpose of the present paper.

The problem of separation of variables and the subsequent search for exact solutions in non-static Minkowski spacetimes is of interest in the analysis of quantum effects associated with spin-$\frac{1}{2}$ particles in accelerated frames of reference. It is clear that, for a quantum description of the accelerated particles, it would be desirable to obtain the modes associated with the accelerated system of coordinates. This is possible if the Dirac equation is separable in the corresponding coordinates. Some results for scalar particles have been obtained in this direction (Sanchez 1979, 1981, Sanchez and Whiting 1986) but that is not the case for spinning particles.

This paper is organized as follows. In section 2 the Dirac equation in curvilinear orthogonal coordinates of the form $t=f(u, v)$, $x=g(u, v)$, y, z, is expressed in a diagonal tetrad gauge where the spinor connections become zero and, in passing, the general solution of the massless Dirac equation is obtained in two-dimensional Minkowski spacetimes. In section 3 the separation of variables in the Dirac equation in the coordinates of the form described in section 2 is carried out. In section 4, using the results obtained in section 3, exact solutions to the Dirac equation in Minkowski curvilinear coordinates are obtained.

2. Dirac equation in curvilinear coordinates

Let us consider the system of curvilinear coordinates

$$t=f(u, v) \qquad x=g(u, v) \qquad y \qquad z \tag{2.1}$$

where the functions f and g satisfy the relations

$$g_{,u}=f_{,v} \qquad g_{,v}=f_{,u}. \tag{2.2}$$

Then, the line element expressed in the coordinates u, v takes the form

$$ds^2=(f_{,u}^2-f_{,v}^2)(dv^2-du^2)+dy^2+dz^2. \tag{2.3}$$

The covariant generalization of the Dirac equation reads

$$(\bar{\gamma}^{\nu}(\partial_{\nu}-\Gamma_{\nu})+m)\Psi=0 \tag{2.4}$$

where the gamma matrices $\bar{\gamma}^{\nu}$ are related to the constant gamma matrices γ^{μ} as follows:

$$\bar{\gamma}^{\nu}=h^{\nu}{}_{\mu}\gamma^{\mu} \qquad \gamma^{\mu}=h_{\nu}{}^{\mu}\bar{\gamma}^{\nu} \tag{2.5}$$

with

$$[\bar{\gamma}^{\mu}, \bar{\gamma}^{\nu}]_{+}=2g^{\mu\nu} \qquad g_{\mu\nu}=\mathrm{diag}(-(f_{,u}^2-f_{,v}^2), (f_{,u}^2-f_{,v}^2), 1, 1) \tag{2.6}$$

$$[\gamma^{\mu}, \gamma^{\nu}]_{+}=2\eta^{\mu\nu} \qquad \eta_{\mu\nu}=\mathrm{diag}(-1, 1, 1, 1) \tag{2.7}$$

where the spinor connections Γ_λ are given by (Brill and Wheeler 1957)

$$\Gamma_\lambda = \tfrac{1}{4} g_{\mu\alpha}(\partial_\lambda h_\nu{}^{\mathcal{H}} h^\alpha{}_{\mathcal{H}} - \Gamma^\alpha{}_{\nu\lambda}) s^{\mu\nu} \tag{2.8}$$

with

$$s^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu). \tag{2.9}$$

From (2.5) and (2.6) it is clear that two different representations of the Dirac matrices are related through a similarity relation and, therefore, there is an infinite set of γ-matrices to write (2.4). Among the possible Dirac matrix representations, we choose to work in two $h^\mu{}_\nu$ gauges which have a simple geometrical interpretation: the diagonal (rotating) gauge tetrad, where the curvilinear $\bar{\gamma}$ matrices are related to the constant γ by

$$\bar{\gamma}^u = \frac{1}{h}\gamma^0 \qquad \bar{\gamma}^v = \frac{1}{h}\gamma^1 \qquad \bar{\gamma}^y = \gamma^2 \qquad \bar{\gamma}^z = \gamma^3 \tag{2.10}$$

where

$$h = (f_{,u}^2 - f_{,v}^2)^{1/2} \tag{2.11}$$

and the Cartesian (fixed) gauge tetrad, where the Dirac matrices $\tilde{\gamma}$, which also satisfy the anticommutation relations (2.7), are related to the constant γ matrices by

$$\begin{aligned} &\tilde{\gamma}^u = h^{-1}(\gamma^0 f_{,u} - \gamma^1 f_{,v}) \qquad \tilde{\gamma}^v = h^{-1}(\gamma^1 f_{,u} - \gamma^0 f_{,v}) \\ &\tilde{\gamma}^y = \gamma^2 \qquad \tilde{\gamma}^z = \gamma^3. \end{aligned} \tag{2.12}$$

In the above gauge the Dirac equation takes the simple form

$$\left(\frac{\tilde{\gamma}^u}{h}\partial_u + \frac{\tilde{\gamma}^v}{h}\partial_v + \gamma^2\partial_y + \gamma^3\partial_z + m\right)\Psi_c = 0 \tag{2.13}$$

and no spinor connections are present.

The transformation S, relating the Dirac matrices in the Cartesian gauge and the constant γ matrices,

$$S^{-1}\tilde{\gamma}^\nu S = \gamma^\nu \tag{2.14}$$

takes the form

$$S = \cosh\frac{\vartheta}{2} - \gamma^0\gamma^1 \sinh\frac{\vartheta}{2} = \exp - \frac{\vartheta}{2}\gamma^0\gamma^1 \tag{2.15}$$

where the value of ϑ is given by the expression

$$\tanh\vartheta = \frac{f_{,v}}{f_{,u}}. \tag{2.16}$$

Then, using the matrix transformation S it is possible, from (2.14), to obtain the corresponding Dirac equation in the diagonal gauge. Substituting (2.14) into (2.13) the Dirac equation can then be written as

$$\left(\frac{\gamma^0}{h}\partial_u + \frac{\gamma^1}{h}\partial_v + \gamma^2\partial_y + \gamma^3\partial_z + m\right)\Phi = 0 \tag{2.17}$$

where

$$\Psi_c = h^{-1/2} S\Phi. \tag{2.18}$$

Equation (2.17) takes a particularly simple form when we consider a massless Dirac particle in a two-dimensional spacetime ($k_y = k_z = 0$). In this case, due to the conformal invariance of the resulting equation, the Dirac equation takes the form

$$(\gamma^0 \partial_u + \gamma^1 \partial_v)\Phi = 0 \tag{2.19}$$

and using the representation

$$\gamma^0 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \tag{2.20}$$

for the Dirac matrices the solution of (2.13) reads

$$\Psi_c = (f_{,u} - f_{,v})^{-1/2} \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} e^{i\omega(u-v)} \tag{2.21}$$

being the above expression solution of the massless Dirac equation for any two-dimensional line element of the form

$$ds^2 = (f_{,u}^2 - f_{,v}^2)(dv^2 - du^2). \tag{2.22}$$

3. Separation of variables

In order to separate variables in the Dirac equation expressed in the coordinates (2.1) we choose to work with (2.17). The reasons for such a selection become evident if we look at the simple expression that the Dirac equation takes in the diagonal tetrad gauge when we make the transformation that reduces the spinor connections to zero. Then, applying the method of separation of variables proposed by Shishkin (1976), we are able to write the Dirac equation (2.17) as a sum of two commuting first-order differential operators:

$$(\hat{K}_1 + \hat{K}_2)\Sigma = 0 \qquad [\hat{K}_1, \hat{K}_2]_- = 0 \tag{3.1}$$

with

$$\Phi = \gamma^2 \gamma^3 \Sigma \tag{3.2}$$

$$\hat{K}_2 \Sigma = -\hat{K}_1 \Sigma = i\ell \Sigma \qquad \Sigma = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} \tag{3.3}$$

$$\hat{K}_1 = \frac{1}{h}(\gamma^0 \partial_u + \gamma^1 \partial_v)\gamma^2 \gamma^3 \tag{3.4}$$

$$\hat{K}_2 = (\gamma^2 \partial_y + \gamma^3 \partial_z)\gamma^2 \gamma^3. \tag{3.5}$$

Noticing that the operator (3.5) commutes with linear momenta operators $-i\partial_y$, $-i\partial_z$, with eigenvalues k_y and k_z respectively, in the representation for the Dirac matrices (2.20) the spinor Σ then takes the form

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ \dfrac{im - ik_y}{k_z + \ell} \sigma^3 \Sigma_1 \end{pmatrix} \exp i(k_y y + k_z z) \tag{3.6}$$

with

$$\kappa^2 = k_y^2 + k_z^2 + m^2 \tag{3.7}$$

where Σ_1 satisfies the equation

$$(\sigma^1\partial_0 + \mathrm{i}\sigma^2\partial_1 + \mathrm{i}\kappa h)\Sigma_1 = 0. \tag{3.8}$$

Equation (3.8) cannot be separated in terms of two commuting first-order differential operators if the function h depends on the two variables u and v. As a preliminary step in the process of separation of variables, we introduce a further similarity transformation T acting on the σ-matrices and on the spinor Σ_1:

$$T = \mathrm{e}^{\alpha} \exp(\mathrm{i}\beta\sigma^3) \tag{3.9}$$

with $\alpha(u, v)$ and $\beta(u, v)$.

Applying transformation (3.9) to (3.8), and imposing the conditions

$$\mathrm{i}\beta_{,u} = \alpha_{,v} \qquad \mathrm{i}\beta_{,v} = \alpha_{,u} \tag{3.10}$$

we obtain

$$[\sigma^1\partial_u + \mathrm{i}\sigma^2\partial_v + \mathrm{i}\kappa h \exp(2\mathrm{i}\beta\sigma^3)]Y = 0 \tag{3.11}$$

where the spinor Y is related to Σ by the expression

$$TY = \Sigma \qquad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}. \tag{3.12}$$

In order to separate variables in (3.11) we impose the relation

$$\exp(2\mathrm{i}\beta\sigma^3) = \frac{a(u) + \mathrm{i}b(v)\sigma^3}{h}. \tag{3.13}$$

Then, the differential equation for the spinor Y takes the form

$$(L_1 - L_2\sigma^3)Y = 0 \tag{3.14}$$

$$L_1 = \sigma^1\partial_u + \mathrm{i}\kappa a(u) \tag{3.15}$$

$$L_2 = \sigma^1\partial_v + \kappa b(v). \tag{3.16}$$

Introducing the auxiliary spinor W,

$$Y = (\sigma^3 L_1 + L_2)W \tag{3.17}$$

we obtain a separable second-order differential equation for the spinor W. Introducing the constant of separability λ and the spinor Δ, the resulting system of equations reads

$$[\partial_u^2 + \kappa^2 a^2 + \mathrm{i}\kappa\sigma^3 a_{,u}]\Delta = \lambda^2\Delta \tag{3.18}$$

$$[\partial_v^2 - \kappa^2 b^2 + \kappa\sigma^3 b_{,v}]\Delta = \lambda^2\Delta \tag{3.19}$$

where Δ is related to W by the transformation

$$W = Q\Delta \qquad Q = \frac{1}{\sqrt{2}}(1 - \mathrm{i}\sigma^2). \tag{3.20}$$

Notice that, due to the explicit form of the Pauli matrix σ^3, (3.18) and (3.19) are decoupled and the spinor Δ can be written in the simple form

$$\Delta = \begin{pmatrix} \xi(u)A(v) \\ \zeta(u)B(v) \end{pmatrix} \tag{3.21}$$

where the functions ξ, ζ, A and B satisfy the two systems of coupled differential equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}+\mathrm{i}\not{k}a(u)\right)\xi=\lambda\zeta \tag{3.22}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}-\mathrm{i}\not{k}a(u)\right)\zeta=\lambda\xi \tag{3.23}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}v}+\not{k}b(v)\right)A=\lambda B \tag{3.24}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}v}-\not{k}b(v)\right)B=\lambda A. \tag{3.25}$$

Substituting Δ into (42) and using expression (3.17) Y is given by

$$Y=\sqrt{2}\,\lambda\begin{pmatrix}\xi B+\zeta A\\ \xi B-\zeta A\end{pmatrix}. \tag{3.26}$$

The above expression for Y can be substituted into (3.12), and, using (3.6) with (3.2), the solution to the Dirac equation (2.13) is then

$$\Psi_{\mathrm{c}}=h^{-1/2}S\begin{pmatrix}\dfrac{\mathrm{i}m-\not{k}_y}{k_z+\not{k}}TY\\ -\sigma^3TY\end{pmatrix}\exp\mathrm{i}(k_yy+k_zz). \tag{3.27}$$

It should be mentioned that condition (3.13) implies that Lame's function presents an additive character of the form

$$h^2=a^2(u)+b^2(v). \tag{3.28}$$

Condition (3.28), for the Lame function h, is a necessary condition of separability for the Klein–Gordon equation in two-dimensional orthogonal curvilinear coordinates (Kalnins 1975, Miller 1977). Therefore our search for orthogonal systems of coordinates where the Dirac equation admits a complete separation of variables reduces to those where the Klein–Gordon equation is also separable.

4. Exact solutions

In this section we shall obtain exact solutions to the Dirac equation (2.13) or (2.17) in the curvilinear orthogonal coordinates where the separation of variables is possible, i.e. there exist functions $a(u)$ and $b(v)$ satisfying condition (3.28) with

$$\beta_{,uu}-\beta_{,vv}=0 \tag{4.1}$$

where β is given by the expression

$$\beta=\tfrac{1}{2}\tan^{-1}\frac{b(v)}{a(u)}. \tag{4.2}$$

Obviously the Dirac equation is separable also in the simple case when h depends on only one variable. This situation corresponds to the following system of coordinates (hyperbolic coordinates):

$$t=\mathrm{e}^{v}\sinh u\qquad x=\mathrm{e}^{v}\cosh u\qquad y\qquad z \tag{4.3}$$

with a line element of the form

$$ds^2 = e^{2v}(dv^2 - du^2) + dy^2 + dz^2. \tag{4.4}$$

For this metric the h-function is $h = e^v$. Then, using the explicit representation of the Pauli matrices, the equation

$$(\sigma^1\partial_u + i\sigma^2\partial_v + ik\, e^v)\Sigma_1 = 0 \tag{4.5}$$

can be rewritten as follows:

$$\left(\frac{d}{dv} + k\, e^v\right)\Sigma_1^1 = -\omega\Sigma_1^2 \tag{4.6}$$

$$\left(\frac{d}{dv} - k\, e^v\right)\Sigma_1^2 = \omega\Sigma_1^1 \tag{4.7}$$

where we have considered

$$\Sigma_1 = \begin{pmatrix}\Sigma_1^1\\ \Sigma_1^2\end{pmatrix} e^{i\omega u} \qquad \partial_u\Sigma_1 = i\omega\Sigma_1. \tag{4.8}$$

The solution of the system (4.6), (4.7) is

$$\Sigma_1^1 = e^{-k\, e^v}[c_1(2k\, e^v)^{i\omega}M(i\omega, 2i\omega + 1, 2k\, e^v) + c_2(2k\, e^v)^{-i\omega}M(-i\omega, -2i\omega + 1, 2k\, e^v)] \tag{4.8}$$

$$\Sigma_1^2 = -i\, e^{-k\, e^v}[c_1(2k\, e^v)^{i\omega}M(i\omega, 2i\omega + 1, 2k\, e^v) - c_2(2k\, e^v)^{-i\omega}M(-i\omega, -2i\omega + 1, 2k\, e^v)] \tag{4.10}$$

where $M(a, b, x)$ is the confluent hypergeometric function (Abramowitz and Stegun 1964). The spinor Ψ_c solution of (2.13) is related to Σ (3.6) as follows:

$$\Psi_c = e^{-v/2}\exp\left[-\left(\frac{u}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma. \tag{4.11}$$

Now, we proceed to solve the Dirac equation in the systems of coordinates where it is possible to find functions β of the form (4.2) satisfying condition (4.1).

4.1. *Parabolic cylinder coordinates (parabolic coordinates of type I) (Kalnins 1975)*

$$t = \tfrac{1}{2}(u^2 + v^2) \qquad x = uv. \tag{4.12}$$

For this system of coordinates the line element takes the form

$$ds^2 = (u^2 - v^2)(dv^2 - du^2) + dy^2 + dz^2. \tag{4.13}$$

The spinor Ψ_c solution of (4.13) is related to Σ as follows:

$$\Psi_c = (u^2 - v^2)^{-1/4}\exp\left[-\left(\tfrac{1}{2}\tanh^{-1}\frac{v}{u}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.14}$$

and the functions $a(u)$ and $b(v)$ in (3.13) take the form

$$a(u) = u \qquad b(v) = -iv. \tag{4.15}$$

Then, the matrix transformation T (3.9) reads

$$T = (u^2 - v^2)^{1/2}\exp\left(\tfrac{1}{2}\tanh^{-1}\frac{v}{u}\sigma^3\right) = \begin{pmatrix}(u-v)^{1/2} & 0\\ 0 & (u+v)^{1/2}\end{pmatrix}. \tag{4.16}$$

Substituting the value of $a(u)$ given in (4.15) into the coupled system of ordinary differential equations (3.22), (3.23) then ζ and ξ reduce to solutions of two parabolic cylinder equations (Abramowitz and Stegun 1964), taking the general form

$$\zeta=\exp\left[-\left(\frac{k}{2}u^2\right)\right]\left(d_1 M\left(\frac{\lambda^2}{4k},\tfrac{1}{2},ku^2\right)+d_2 vM\left(\frac{\lambda^2}{4k}+1,\tfrac{3}{2},ku^2\right)\right) \tag{4.17}$$

$$\xi=\exp\left[-\left(\frac{k}{2}u^2\right)\right]\left(\frac{d_2}{\lambda}M\left(\frac{\lambda^2}{4k}+\tfrac{1}{2},\tfrac{1}{2},ku^2\right)+d_1\lambda vM\left(\frac{\lambda^2}{4k}+1,\tfrac{3}{2},ku^2\right)\right) \tag{4.18}$$

where d_1, d_2 are arbitrary constants.

Analogously, the functions $A(v)$ and $B(v)$, solutions of the system (3.24), (3.25), take the form

$$A=\exp\left[-\left(\frac{k}{2}v^2\right)\right]\left(c_1 M\left(\frac{\lambda^2}{4k},\tfrac{1}{2},kv^2\right)+c_2 vM\left(\frac{\lambda^2}{4k}+1,\tfrac{3}{2},kv^2\right)\right) \tag{4.19}$$

$$B=\exp\left[-\left(\frac{k}{2}v^2\right)\right]\left(\frac{c_2}{\lambda}M\left(\frac{\lambda^2}{4k}+\tfrac{1}{2},\tfrac{1}{2},kv^2\right)+c_1\lambda vM\left(\frac{\lambda^2}{4k}+1,\tfrac{3}{2},kv^2\right)\right) \tag{4.20}$$

where c_1 and c_2 are constants.

4.2. Hyperbolic coordinates of type II

$$\begin{aligned} t&=\tfrac{1}{2}\sinh(u-v)+\exp(u+v)\\ x&=-\tfrac{1}{2}\sinh(u-v)+\exp(u+v) \qquad y \qquad z. \end{aligned} \tag{4.21}$$

The line element can be written as

$$\mathrm{d}s^2=(\mathrm{e}^{2u}+\mathrm{e}^{2v})(\mathrm{d}v^2-\mathrm{d}u^2)+\mathrm{d}y^2+\mathrm{d}z^2. \tag{4.22}$$

For the coordinates (4.21), the spinor Ψ_c solution of the Dirac equation (2.13) is related to Σ (3.6) as follows:

$$\Psi_c=(\mathrm{e}^{2u}+\mathrm{e}^{2v})^{-1/4}\exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.23}$$

where the 'angle' ϑ is determined by the relation

$$\tanh\vartheta=\frac{2\,\mathrm{e}^{u+v}-\cosh(u-v)}{2\,\mathrm{e}^{u+v}+\cosh(u-v)}. \tag{4.24}$$

For the present system of coordinates, the functions $a(u)$ and $b(v)$, allowing the separation of variables in the Dirac equation, are

$$a=\mathrm{e}^u \qquad b=\mathrm{e}^v. \tag{4.25}$$

Therefore, the functions α and β take the form

$$\beta=\tfrac{1}{2}\tan^{-1}\mathrm{e}^{v-u} \qquad \alpha=-\frac{\mathrm{i}}{2}\tan^{-1}\mathrm{e}^{v}. \tag{4.26}$$

From (4.26) the matrix transformation T is given by the expression

$$T=\begin{pmatrix}1 & 0\\ 0 & (\mathrm{e}^u-\mathrm{i}\mathrm{e}^v)/(\mathrm{e}^{2u}+\mathrm{e}^{2v})^{1/2}\end{pmatrix}. \tag{4.27}$$

After substituting the value of $a(u)$ into the system of equations (3.22), (3.23) we find that the functions ξ, ζ take the form

$$\xi = \exp[-(\mathrm{i}k\,\mathrm{e}^u)][b_1(2\mathrm{i}k\,\mathrm{e}^u)^\lambda M(\lambda, 2\lambda+1, 2\mathrm{i}k\,\mathrm{e}^u) + b_2(2\mathrm{i}k\,\mathrm{e}^u)^{-\lambda} \times M(-\lambda, -2\lambda+1, 2\mathrm{i}k\,\mathrm{e}^u)] \tag{4.28}$$

$$\zeta = \exp[-(\mathrm{i}k\,\mathrm{e}^u)][b_1(2k\,\mathrm{e}^u)^\lambda M(\lambda+1, 2\lambda+1, 2\mathrm{i}k\,\mathrm{e}^u) - b_2(2\mathrm{i}k\,\mathrm{e}^u)^{-\lambda} \times M(-\lambda+1, -2\lambda+1, 2\mathrm{i}k\,\mathrm{e}^u)] \tag{4.29}$$

where b_1 and b_2 are constants.

Analogously, the solution of the system (3.24), (3.25) with $b(v) = \mathrm{e}^v$ is

$$A = \exp[-(k\,\mathrm{e}^v)][a_1(2k\,\mathrm{e}^v)^\lambda M(\lambda, 2\lambda+1, 2k\,\mathrm{e}^v) + a_2(2k\,\mathrm{e}^v)^{-\lambda} \times M(-\lambda, -2\lambda+1, 2k\,\mathrm{e}^v)] \tag{4.30}$$

$$B = \exp[-(k\,\mathrm{e}^v)][a_1(2k\,\mathrm{e}^v)^\lambda M(\lambda+1, 2\lambda+1, 2k\,\mathrm{e}^v) - a_2(2k\,\mathrm{e}^v)^{-\lambda} \times M(-\lambda+1, -2\lambda+1, 2k\,\mathrm{e}^v)] \tag{4.31}$$

where a_1 and a_2 are arbitrary constants.

4.3. *Hyperbolic coordinates of type III*

$$t = \tfrac{1}{2}\cosh(u-v) + \mathrm{e}^{u+v} \qquad x = -\tfrac{1}{2}\cosh(u-v) + \mathrm{e}^{u+v} \qquad y \qquad z. \tag{4.32}$$

The line element associated with the coordinates (4.32) takes the form

$$\mathrm{d}s^2 = (\mathrm{e}^{2u} - \mathrm{e}^{2v})(\mathrm{d}v^2 - \mathrm{d}u^2) + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{4.33}$$

and the solution Ψ_c of the Dirac equation is related to Σ (3.6) as follows:

$$\Psi = (\mathrm{e}^{2u} - \mathrm{e}^{2v})^{-1/4} \exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.34}$$

where ϑ is

$$\tanh\vartheta = \frac{2\,\mathrm{e}^{u+v} - \sinh(u-v)}{2\,\mathrm{e}^{u+v} + \sinh(u-v)}. \tag{4.35}$$

For this hyperbolic system of coordinates, the values of $a(u)$ and $b(v)$ are

$$a = \mathrm{e}^u \qquad \mathrm{i}b = \mathrm{e}^v. \tag{4.36}$$

Then, the corresponding values of α and β read

$$\mathrm{i}\beta = \tfrac{1}{2}\tanh^{-1}\mathrm{e}^{v-u} \qquad \alpha = -\tfrac{1}{2}\tanh^{-1}\mathrm{e}^{v-u} \tag{4.37}$$

and the matrix T takes the form

$$T = \begin{pmatrix} 1 & 0 \\ 0 & (\mathrm{e}^u - \mathrm{e}^v)/(\mathrm{e}^{2u} + \mathrm{e}^{2v})^{1/2} \end{pmatrix}. \tag{4.38}$$

From (4.36) and (4.25) we see that the u dependence of the spinor Σ (3.6) for the systems of coordinates (4.32) and (4.21) coincide, therefore the functions ξ and ζ, solutions of the system (3.22), (3.23) for the hyperbolic coordinates of type III (4.32), are given by (4.28), (4.29).

Analogously, the solution of the system (3.24), (3.25) can be obtained in a straightforward way, giving the result

$$A=\exp[-(\mathrm{i}k\,\mathrm{e}^{v})][a_1(2\mathrm{i}k\,\mathrm{e}^{v})^{\lambda}M(\lambda+1,2\lambda+1,2\mathrm{i}k\,\mathrm{e}^{v})-a_2(2\mathrm{i}k\,\mathrm{e}^{v})^{-\lambda}\times M(-\lambda+1,-2\lambda+1,2\mathrm{i}k\,\mathrm{e}^{v})] \tag{4.39}$$

$$B=\exp[-(\mathrm{i}k\,\mathrm{e}^{v})][a_1(2\mathrm{i}k\,\mathrm{e}^{v})^{\lambda}M(\lambda,2\lambda+1,2\mathrm{i}k\,\mathrm{e}^{v})+a_2(2\mathrm{i}k\,\mathrm{e}^{v})^{-\lambda}\times M(-\lambda,-2\lambda+1,2\mathrm{i}k\,\mathrm{e}^{v})]. \tag{4.40}$$

4.4. *Elliptic coordinates of type II*

(i) $$t=\cos u\cos v \qquad x=-\sin u\sin v \qquad y \qquad z. \tag{4.41}$$

The interval associated with this system of coordinates reads

$$\mathrm{d}s^2=(\sin^2 u-\sin^2 v)(\mathrm{d}v^2-\mathrm{d}u^2)+\mathrm{d}y^2+\mathrm{d}z^2. \tag{4.42}$$

The spinor solution to the Dirac equation Ψ_c in the Cartesian gauge is

$$\Psi_c=(\sin^2 u-\sin^2 v)^{-1/4}\exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.43}$$

where the general form of Σ is given by (3.6) and

$$\tanh\vartheta=\coth u\tan v. \tag{4.44}$$

The functions $a(u)$ and $b(v)$, allowing the complete separation of variables in the Dirac equation in the coordinates (4.41), are

$$a=\sin u \qquad \mathrm{i}b=\sin v. \tag{4.45}$$

Substituting (4.45) into (3.13) and using (3.10) we obtain

$$2\mathrm{i}\beta=\tanh^{-1}\frac{\sin v}{\sin u} \qquad 2\alpha=-\tanh^{-1}\frac{\cos u}{\cos v} \tag{4.46}$$

and the matrix T reads

$$T=(\cos u+\cos v)^{-1/2}\begin{pmatrix}(\sin u+\sin v)^{1/2} & 0\\ 0 & (\sin u-\sin v)^{1/2}\end{pmatrix}. \tag{4.47}$$

Substituting $a(u)=\sin u$ into (3.22), (3.23) we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}+\mathrm{i}k\sin u\right)\xi=\lambda\zeta \tag{4.48}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}-\mathrm{i}k\sin u\right)\zeta=\lambda\xi. \tag{4.49}$$

Making the ansatz

$$\xi=\mathrm{e}^{\mathrm{i}k\cos u}V_1+\mathrm{e}^{-\mathrm{i}k\cos u}V_2 \tag{4.50}$$

$$\zeta=\mathrm{e}^{\mathrm{i}k\cos u}Z_1+\mathrm{e}^{-\mathrm{i}k\cos u}Z_2 \tag{4.51}$$

we obtain an equivalent system of equations for the new functions V_1, V_2 and Z_1, Z_2:

$$\frac{\mathrm{d}}{\mathrm{d}u} V_1 = \lambda Z_1 \tag{4.52}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} V_2 + 2\mathrm{i} \sin u\, V_2 = \lambda Z_2 \tag{4.53}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} Z_1 = \lambda Y_1 \tag{4.54}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} Z_1 - 2\mathrm{i} \sin u\, Z_1 = \lambda Y_1 . \tag{4.55}$$

Substituting (4.52) into (4.55) and making the change of variable $u = 2\vartheta$, we obtain the following Ince's equation (Arscott 1967) for the function V_1:

$$\ddot{V}_1 - 4\mathrm{i} \sin 2\vartheta\, \dot{V}_1 - 4\lambda^2 V_1 = 0 \tag{4.56}$$

where the dot indicates derivation with respect to the variable ϑ. Analogously, from (4.53) and (4.54) we obtain

$$\ddot{Z}_2 + 4\mathrm{i} \sin 2\vartheta\, \dot{Z}_2 - 4\lambda^2 Z_2 = 0. \tag{4.57}$$

The solutions of (4.56) and (4.57) can be expressed in terms of the convergent series

$$V_1 = \sum_0^\infty C_{2r} \cos 2r\vartheta \tag{4.58}$$

with

$$\begin{aligned} &\lambda^2 C_0 + \mathrm{i} C_2 = 0 \\ &(1+\lambda^2) C_2 + 2\mathrm{i} C_4 = 0 \\ &-(r-1)\mathrm{i} C_{2r-2} + (1+\lambda^2) C_{2r} + (r+1)\mathrm{i} C_{2r+1} = 0 \qquad (r \geqslant 2) \end{aligned} \tag{4.59}$$

and

$$Z_2 = \sum_0^\infty A_{2r} \cos 2r\vartheta \tag{4.60}$$

with

$$\begin{aligned} &\lambda^2 A_0 - \mathrm{i} A_2 = 0 \\ &(1+\lambda^2) A_2 - 2\mathrm{i} A_4 = 0 \\ &(r-1)\mathrm{i} A_{2r-2} + (1+\lambda^2) A_{2r} - (r+1)\mathrm{i} A_{2r+1} = 0 \qquad (r \geqslant 2). \end{aligned} \tag{4.61}$$

Then, the solution of the system of equations reads

$$\xi = \mathrm{e}^{\mathrm{i}\ell \cos u} \left(\sum_0^\infty C_n \cos nu \right) + \mathrm{e}^{-\mathrm{i}\ell \cos u} \left(-\frac{1}{2\lambda} \sum_0^\infty n A_n \sin nu \right) \tag{4.62}$$

$$\zeta = \mathrm{e}^{\mathrm{i}\ell \cos u} \left(-\frac{1}{2\lambda} \sum_0^\infty n C_n \sin nu \right) + \mathrm{e}^{-\mathrm{i}\ell \cos u} \left(\sum_0^\infty A_n \cos nu \right) \tag{4.63}$$

where we have made the substitution $n = 2r$.

Following the results obtained by Urwin and Arscott (1969) the series (4.58) and (4.60) are convergent for values of u and v. The solution of the system (3.24), (3.25) when $b(v)$ is given by (4.45) can be obtained in a straightforward way making the changes $k\to -k$ and $u\to v$. Then we have

$$A=e^{-ik\cos v}\left(\sum_0^\infty C_n\cos nv\right)+e^{ik\cos v}\left(-\frac{1}{2\lambda}\sum_0^\infty nA_n\sin nv\right) \tag{4.64}$$

$$B=e^{-ik\cos v}\left(-\frac{1}{2\lambda}\sum_0^\infty nC_n\sin nv\right)+e^{ik\cos v}\left(\sum_0^\infty A_n\cos nv\right). \tag{4.65}$$

(ii) $$t=\cosh u\cosh v \qquad x=\sinh u\sinh v \qquad y \qquad z. \tag{4.66}$$

The line element in this case is

$$ds^2=(\cosh^2 u-\cosh^2 v)(dv^2-du^2)+dy^2+dz^2 \tag{4.67}$$

and the spinor Ψ_c is related to Σ as follows:

$$\Psi_c=(\cosh^2 u-\cosh^2 v)^{-1/4}\exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.68}$$

where ϑ is given by the relation

$$\tanh\vartheta=\tanh v\coth u. \tag{4.69}$$

The functions $a(u)$ and $b(v)$, allowing the complete separation of variables in the Dirac equation, are

$$a=\sinh u \qquad ib=\sinh v \tag{4.70}$$

and the corresponding β- and α-functions take the form

$$i\beta=\tfrac{1}{2}\tanh^{-1}\frac{\sinh v}{\sinh u} \qquad \alpha=\tfrac{1}{2}\tanh^{-1}\frac{\cosh v}{\cosh u}. \tag{4.71}$$

Substituting (4.71) into (3.9) we obtain

$$T=(\cosh u+\cosh v)^{-1/2}\begin{pmatrix}(\sinh u-\sinh v)^{1/2} & 0\\ 0 & (\sinh u+\sinh v)^{1/2}\end{pmatrix}. \tag{4.72}$$

Also, substituting the value of $a(u)$, given by (4.69), into the system (3.22), (3.23), we obtain

$$\left(\frac{d}{du}+ik\sinh u\right)\xi=\lambda\zeta \tag{4.73}$$

$$\left(\frac{d}{du}-ik\sinh u\right)\zeta=\lambda\xi. \tag{4.74}$$

Noticing that (4.73), (4.74) reduces to the system (4.48), (4.49) when we make the changes $\lambda\to i\lambda$, $u\to iu$, $k\to -k$, the solution of the above system of equations (4.73), (4.74) is then

$$\xi=e^{-ik\cosh u}\left(\sum_0^\infty \tilde{C}_n\cosh nu\right)+e^{ik\cos u}\left(-\frac{i}{2\lambda}\sum_0^\infty n\tilde{A}_n\sinh nu\right) \tag{4.75}$$

$$\zeta=e^{-ik\cosh u}\left(-\frac{i}{2\lambda}\sum_0^\infty n\tilde{C}_n\sinh nu\right)+e^{ik\cosh u}\left(\sum_0^\infty \tilde{A}_n\cosh nu\right) \tag{4.76}$$

where the coefficients $\tilde{C}_n$ and $\tilde{A}_n$, after the change $\lambda \to i\lambda$, satisfy the recurrence relations (4.59) and (4.61), respectively. Analogously, the solution of the system (3.24), (3.25) with $b(v) = -i \sinh v$ can be obtained from (4.64), (4.65) in a simple way, giving

$$A = e^{i\ell \cosh v}\left(\sum_0^\infty \tilde{C}_n \cosh nv\right) + e^{-i\ell \cosh v}\left(-\frac{i}{2\lambda}\sum_0^\infty n\tilde{A}_n \sinh nv\right) \tag{4.77}$$

$$B = e^{i\ell \cosh v}\left(-\frac{i}{2\lambda}\sum_0^\infty n\tilde{C}_n \sinh nv\right) + e^{-i\ell \cosh v}\left(\sum_0^\infty \tilde{A}_n \cosh nv\right). \tag{4.78}$$

4.5. *Elliptic coordinates of type I*

$$t = \sinh u \cosh v \qquad x = \cosh u \sinh v \qquad y \qquad z. \tag{4.79}$$

The associated line element is

$$ds^2 = (\sinh^2 u + \cosh^2 v)(dv^2 - du^2) + dy^2 + dz^2 \tag{4.80}$$

and the spinor Ψ_c is related to Σ as follows:

$$\Psi_c = (\sinh^2 u + \cosh^2 v)^{-1/4} \exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.81}$$

with

$$\tanh \vartheta = \tanh u \tanh v. \tag{4.82}$$

The functions $a(u)$ and $b(v)$ are given by the expressions

$$a = \sinh u \qquad b = \cosh v \tag{4.83}$$

and the α- and β-functions take the form

$$\beta = \tfrac{1}{2}\tan^{-1}\frac{\cosh v}{\sinh u} \qquad \alpha = -\frac{i}{2}\tan^{-1}\frac{\sinh v}{\cosh u}. \tag{4.84}$$

Substituting (4.84) into (3.9) we obtain

$$T = (\cosh u + i \sinh v)^{-1/2}\begin{pmatrix} (\sinh u + i\cosh v)^{1/2} & 0 \\ 0 & (\sinh u - i \cosh v)^{1/2} \end{pmatrix}. \tag{4.85}$$

The value of $a(u)$ in (4.83) coincides with the corresponding value obtained for $a(u)$ in the elliptic system of coordinates (4.66). Then, for (4.79), ξ and ζ are given by (4.75) and (4.76), respectively. The functions $A(v)$ and $B(v)$, solutions of the system (3.24), (3.25) $b(v) = \cosh v$, can be obtained from (4.77), (4.78), making the change $\lambda \to i\lambda$ and $v \to iv - \pi/2$. Then, we have

$$A = e^{-\ell \sinh v}\left(i\sum_0^\infty \tilde{C}_n \sinh nv\right) + e^{\ell \sinh v}\left(\frac{1}{2\lambda}\sum_0^\infty n\tilde{A}_n \cosh nv\right) \tag{4.86}$$

$$B = e^{-\ell \sinh v}\left(\frac{1}{2\lambda}\sum_0^\infty n\tilde{C}_n \cosh nv\right) + e^{\ell \sinh v}\left(i\sum_0^\infty \tilde{A}_n \sinh nv\right). \tag{4.87}$$

The systems of coordinates (4.3), (4.12), (4.21), (4.32), (4.41), (4.66), (4.79) with the Cartesian coordinates are the eight non-equivalent system of coordinates where the Dirac equation separates using the second-order formalism proposed in section 3. Now, we proceed, for the sake of completeness, to exhibit the transformation S (2.18) relating the Dirac spinor in the Cartesian and rotating gauges in the remaining two orthogonal systems of coordinates where the Klein–Gordon allows a complete separation of variables and the Dirac equation is not separable.

4.6. Parabolic coordinates of type II

$$t=\tfrac{1}{2}(u-v)^2+(u+v) \qquad x=-\tfrac{1}{2}(u-v)^2+(u+v) \qquad y \qquad z. \tag{4.88}$$

The line element for the coordinates (4.88) is

$$ds^2=4(u-v)(dv^2-du^2)+dy^2+dz^2 \tag{4.89}$$

and the spinor Ψ_c (2.13) is related to Σ as follows:

$$\Psi_c=(u-v)^{-1/4}\exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.90}$$

where the ϑ is given by the relation

$$\tanh\vartheta=\frac{v-u+1}{u-v+1}. \tag{4.91}$$

4.7. Hyperbolic coordinates of type I

$$\begin{aligned} t&=\tfrac{1}{2}[\cosh\tfrac{1}{2}(u-v)+\sinh\tfrac{1}{2}(u+v)] \\ x&=\tfrac{1}{2}[-\cosh\tfrac{1}{2}(u-v)+\sinh\tfrac{1}{2}(u+v)] \qquad y \qquad z. \end{aligned} \tag{4.92}$$

The corresponding line element is

$$ds^2=\tfrac{1}{4}(\sinh u-\sinh v)(dv^2-du^2)+dy^2+dz^2 \tag{4.93}$$

and the spinor Ψ_c is related to Σ as follows:

$$\Psi_c=(\sinh u-\sinh v)^{-1/4}\exp\left[-\left(\frac{\vartheta}{2}\gamma^0\gamma^1\right)\right]\gamma^2\gamma^3\Sigma \tag{4.94}$$

where ϑ is

$$\tanh\vartheta=\frac{\cosh\tfrac{1}{2}(u+v)-\sinh\tfrac{1}{2}(u-v)}{\cosh\tfrac{1}{2}(u+v)+\sinh\tfrac{1}{2}(u-v)}. \tag{4.95}$$

4. Conclusions

The results presented in this paper show that it is possible to separate variables in the Dirac equation in non-static two-dimensional curvilinear coordinates, using a complete set of second-order differential operators. The method of separation applied permits us to obtain the spinor wave solution of the Dirac equation in any tetrad, and does not make use of the Newman–Penrose null tetrad formalism (Chandrasekhar 1983). The exact solutions obtained in the present paper suggest the possibility of investigating quantum effects associated with spinning particles in non-inertial frames of reference. Finally, we note that it is possible to generalize the techniques presented in this paper to analyse other more complicated relativistic wave equations.

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